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A Numerical Method for a Nonlocal Hyperbolic Model Arising from a Reliability System

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Abstract—In this paper, we develop a numerical approximation to the solution of a system of integral equations arising from a nonlocal hyperbolic reliability model. Convergence of this numerical method is proved. And this numerical scheme is used to study the behavior of the solution. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Nonlocal hyperbolic reliability model, System of integral equations, Convergence, Numerical method.

1. INTRODUCTION

In the study of repairable systems, which is an important topic in reliability, there is extensive literature on availability characteristics of repairable systems with two or three components under various assumptions on the failures and repairs; see [1–9] and the references therein. In most of these articles, exponential distributions are assumed for some system variables and only one repair facility is considered for mathematical convenience. Methods used in the existing literature dealing with non-Markov systems involving many general random variables include the *regenerative point technique* (RPT) [1,9,10] and the *supplementary variables method* (SVM) [3–8]. In order to use the RPT, one has to correctly formulate and solve a system of Markov renewal equations, usually using an analytical method which is difficult for a non-Markov repairable system with only a few renewal points. On the other hand, by using the SVM, one can readily obtain all differential equations in terms of the state transition diagram of the model. However, it is

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still not easy to solve these differential equations because they usually involve some functions to be determined if there are at least two hazard rate parameters in one of the equations. In [7], Shi and Li theoretically analyzed a two-unit series reliability system with shut-off rule. They developed a nonlocal hyperbolic model and then reduced it to a system of integral equations. Recently, under certain conditions on the hazard rate parameters, we established the existence and uniqueness results for the system of integral equations [11]. In the present paper, we want to construct a numerical approximation to the solution of the model developed in [7] and use this numerical algorithm to examine the behavior of the model solution.

The paper is organized as follows. In Section 2, we present the model. In Section 3, we develop the numerical approximation and prove its convergence to the unique solution. In Section 4, we numerically study the behavior of the solution.

2. THE MODEL

In their modeling efforts, Shi and Li [7] imposed the following assumptions on the system.

(A1) The system consists of two units in series. The system operates if and only if both units operate.

(A2) Unit 1 upon failure shuts off unit 2, but not vice versa.

(A3) All random variables of the lifetime and the repair time for each unit are mutually independent.

(A4) The lifetime distribution $F_i(t)$ and the repair time distribution $G_i(t)$ of unit i $(i = 1, 2)$ are arbitrary, and they are denoted by

$$\begin{aligned} F_i(t) &= \int_0^t f_i(s)\,ds = 1 - \exp\left(-\int_0^t \lambda_i(s)\,ds\right) = 1 - \bar{F}_i(t), \\ G_i(t) &= \int_0^t g_i(s)\,ds = 1 - \exp\left(-\int_0^t \mu_i(s)\,ds\right) = 1 - \bar{G}_i(t), \end{aligned} \tag{2.1}$$

which implies that for unit i, $f_i(t)$ is the density function of the lifetime, $\lambda_i(t)$ is the hazard rate (a measure of failure rate at time t) of the lifetime, $g_i(t)$ is the density function of the repair time, and $\mu_i(t)$ is the hazard rate of the repair time.

(A5) The repaired unit is as good as a new unit. There is only one repair facility, and the repair discipline for the two units is "first-fail, first-repaired".

Define the system's states as follows.

State 0: two units are operating, and the ages of the two units are equal;
State 01: two units are operating, and the age of unit 1 is greater than that of unit 2;
State 02: two units are operating, and the age of unit 2 is greater than that of unit 1;
State 1: unit 1 is being repaired, and unit 2 is shut off;
State 2: unit 2 is being repaired, and unit 1 is still operating;
State 3: unit 2 is still being repaired, and unit 1 is waiting for repair;
State 4: unit 1 is being repaired, and new unit 2 is in "suspended animation";

and introduce the following notations:

- $S(t)$ is the system state at time t;
- $X(t)$ is the age of unit 1 at time t when $S(t) = 0, 01, 02, 2$;
- $Y(t)$ is the age of unit 2 at time t when $S(t) = 0, 01, 02, 1$;
- $Z_1(t)$ is the elapsed repair time of unit 1 at time t when $S(t) = 1, 4$;
- $Z_2(t)$ is the elapsed repair time of unit 2 at time t when $S(t) = 2, 3$.

Then Shi and Li [7] obtained a Markov process $\{S(t), X(t), Y(t), Z_1(t), Z_2(t)\}$ which takes values on the set

$$J^* = \{(0, x), (01, x, y), (02, x, y), (1, y, z), (2, x, z), (3, z), (4, z) \mid 0 \le x, y, z < \infty\},$$

where x, y refer to ages and z refers to the elapsed repair time.

Defining the following state probabilities:

$$\begin{aligned}
P_0(t,x)\,dx &= P\{S(t)=0,\ x \le X(t) = Y(t) < x+dx\}; \\
P_{01}(t,x,y)\,dx\,dy &= P\{S(t)=0,\ x \le X(t) < x+dx,\ y \le Y(t) < y+dy\}, && x > y; \\
P_{02}(t,x,y)\,dx\,dy &= P\{S(t)=0,\ x \le X(t) < x+dx,\ y \le Y(t) < y+dy\}, && x < y; \\
P_1(t,y,z)\,dz &= P\{S(t)=1,\ z \le Z_1(t) < z+dz,\ Y(t)=y\}; \\
P_2(t,x,z)\,dx\,dz &= P\{S(t)=2,\ x \le X(t) < x+dx,\ z \le Z_2(t) < z+dz\}, && x > z; \\
P_3(t,z)\,dz &= P\{S(t)=3,\ z \le Z_2(t) < z+dz\}; \\
P_4(t,z)\,dz &= P\{S(t)=4,\ z \le Z_1(t) < z+dz\},
\end{aligned}$$

with

$$\begin{gathered}
P_0(x) = \lim_{t\to\infty} P_0(t,x), \qquad P_{01}(x,y) = \lim_{t\to\infty} P_{01}(t,x,y), \qquad P_{02}(x,y) = \lim_{t\to\infty} P_{02}(t,x,y), \\
P_1(y,z) = \lim_{t\to\infty} P_1(t,y,z), \qquad P_2(x,z) = \lim_{t\to\infty} P_2(t,x,z), \qquad P_3(z) = \lim_{t\to\infty} P_3(t,z), \\
P_4(z) = \lim_{t\to\infty} P_4(t,z),
\end{gathered}$$

and using Markov properties, probability considerations, and continuity arguments, they derived the following differential equations governing the behavior of the system:

$$\begin{aligned}
\left[\frac{d}{dx} + \lambda_1(x) + \lambda_2(x)\right] P_0(x) &= 0, \\
\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda_1(x) + \lambda_2(y)\right] P_{01}(x,y) &= 0, \\
\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda_1(x) + \lambda_2(y)\right] P_{02}(x,y) &= 0, \\
\left[\frac{\partial}{\partial z} + \mu_1(z)\right] P_1(y,z) &= 0, \\
\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial z} + \lambda_1(x) + \mu_2(z)\right] P_2(x,z) &= 0, \\
\left[\frac{d}{dz} + \mu_2(z)\right] P_3(z) &= \int_z^\infty P_2(x,z)\lambda_1(x)\,dx, \\
\left[\frac{d}{dz} + \mu_1(z)\right] P_4(z) &= 0,
\end{aligned} \tag{2.2}$$

the boundary conditions

$$\begin{aligned}
P_0(0) &= \int_0^\infty P_4(z)\mu_1(z)\,dz, \\
P_{01}(x,0) &= \int_0^x P_2(x,z)\mu_2(z)\,dz, \\
P_{02}(0,y) &= \int_0^\infty P_1(y,z)\mu_1(z)\,dz, \\
P_1(y,0) &= \int_0^y P_{02}(x,y)\lambda_1(x)\,dx + \int_y^\infty P_{01}(x,y)\lambda_1(x)\,dx + P_0(y)\lambda_1(y), \\
P_2(x,0) &= \int_0^x P_{01}(x,y)\lambda_2(y)dy + \int_x^\infty P_{02}(x,y)\lambda_2(y)dy + P_0(x)\lambda_2(x), \\
P_3(0) &= 0, \\
P_4(0) &= \int_0^\infty P_3(z)\mu_2(z)\,dz,
\end{aligned} \tag{2.3}$$

and the regular condition

$$\begin{aligned}\int_0^\infty P_0(x)\,dx &+ \int_0^\infty \int_0^x P_{01}(x,y)\,dy\,dx + \int_0^\infty \int_0^y P_{02}(x,y)\,dx\,dy \\ + \int_0^\infty \int_0^\infty P_1(y,z)\,dz\,dy &+ \int_0^\infty \int_0^x P_2(x,z)\,dz\,dx + \int_0^\infty [P_3(z)+P_4(z)]\,dz = 1.\end{aligned} \tag{2.4}$$

Under the assumptions that $\lambda_i(t)$ and $\mu_i(t)$ are bounded measurable functions, the nonlocal hyperbolic problem (2.2)–(2.4) was reduced to the following system of integral equations:

$$\begin{aligned} h_1(x) &= \int_0^x h_3(t)g_2(x-t)\,dt, \\ h_2(x) &= \int_0^x h_2(t)f_1(x-t)\,dt + \int_0^\infty h_1(t)f_1(x+t)\,dt + f_1(x), \\ h_3(x) &= \int_0^x h_1(t)f_2(x-t)\,dt + \int_0^\infty h_2(t)f_2(x+t)\,dt + f_2(x). \end{aligned} \tag{2.5}$$

Here the functions h_1, h_2, and h_3 are related to the solution of problem (2.2)–(2.4) in the following manner:

$$\begin{aligned} P_0(x) &= C\bar{F}_1(x)\bar{F}_2(x), \\ P_{01}(x,y) &= Ch_1(x-y)\bar{F}_1(x)\bar{F}_2(y), \\ P_{02}(x,y) &= Ch_2(y-x)\bar{F}_1(x)\bar{F}_2(y), \\ P_1(y,z) &= C\bar{G}_1(z)\bar{F}_2(y)h_2(y), \\ P_2(x,z) &= Ch_3(x-z)\bar{F}_1(x)\bar{G}_2(z), \\ P_3(z) &= C\bar{G}_2(z)\int_0^\infty h_3(t)\left[\bar{F}_1(t)-\bar{F}_1(z+t)\right]\,dt, \\ P_4(z) &= C\bar{G}_1(z), \end{aligned} \tag{2.6}$$

where in view of (2.4) the constant C is determined by

$$\begin{aligned} C^{-1} = \int_0^\infty \bar{F}_1(x)\bar{F}_2(x)\,dx &+ \int_0^\infty \int_0^x h_1(x-y)\bar{F}_1(x)\bar{F}_2(y)\,dy\,dx \\ + \int_0^\infty \int_0^y h_2(y-x)\bar{F}_1(x)\bar{F}_2(y)\,dx\,dy &+ \int_0^\infty \int_0^\infty \bar{G}_1(z)\bar{F}_2(y)h_2(y)\,dz\,dy \\ + \int_0^\infty \int_0^x h_3(x-z)\bar{F}_1(x)\bar{G}_2(z)\,dz\,dx &+ \int_0^\infty \bar{G}_1(z)\,dz \\ + \int_0^\infty \int_0^\infty \bar{G}_2(z)h_3(t)&\left[\bar{F}_1(t)-\bar{F}_1(z+t)\right]\,dt\,dz. \end{aligned} \tag{2.7}$$

It is worth noting that C^{-1} is the mean renewal period of the system. Moreover, the limiting availability of the system, denoted by A, is given by

$$\begin{aligned} A = C\Bigg[\int_0^\infty \bar{F}_1(x)\bar{F}_2(x)\,dx &+ \int_0^\infty \int_0^x h_1(x-y)\bar{F}_1(x)\bar{F}_2(y)\,dy\,dx \\ &+ \int_0^\infty \int_0^y h_2(y-x)\bar{F}_1(x)\bar{F}_2(y)\,dx\,dy\Bigg], \end{aligned} \tag{2.8}$$

and the stationary failure frequency of the system, denoted by W, is given by

$$\begin{aligned} W = C\Bigg\{1 &+ \int_0^\infty \int_0^x h_1(x-y)\left[f_1(x)\bar{F}_2(y)+\bar{F}_1(x)f_2(y)\right]\,dy\,dx \\ &+ \int_0^\infty \int_0^y h_2(y-x)\left[f_1(x)\bar{F}_2(y)+\bar{F}_1(x)f_2(y)\right]\,dx\,dy\Bigg\}. \end{aligned} \tag{2.9}$$

In [11], we proved that if there exist positive constants σ and η such that

$$\sigma \le \lambda_1(x), \lambda_2(x), \mu_2(x) \le \eta, \quad 2\sigma > \eta, \quad \text{and} \quad (2\sigma-\eta)^2(2\sigma+\eta) > \eta^3, \tag{2.10}$$

then system (2.5) has a unique solution.

3. NUMERICAL APPROXIMATION

Throughout this section we assume that λ_i and μ_i are continuous functions satisfying (2.10). Hence, (2.5) has a unique continuously differentiable solution [11]. Our numerical approximation consists of two steps. The first step is to develop a sequence of functions that monotonically converges to the solution of (2.5). The second step involves the discretization of this sequence of functions.

3.1. Monotone Sequence

Consider the following sequence of approximating functions $\{h_1^n, h_2^n, h_3^n\}_{n=0}^{\infty}$ which is analogous to the Gauss-Seidel iterative scheme for solving a system of linear equations. Let

$$h_1^0(x) = 0, \qquad h_2^0(x) = f_1(x), \qquad h_3^0(x) = \int_0^\infty f_1(t) f_2(x+t)\,dt + f_2(x),$$

and for $n = 1, 2, \dots$, let

$$\begin{aligned} h_1^n(x) &= \int_0^x h_3^{n-1}(t) g_2(x-t)\,dt, \\ h_2^n(x) &= \int_0^x h_2^{n-1}(t) f_1(x-t)\,dt + \int_0^\infty h_1^n(t) f_1(x+t)\,dt + f_1(x), \\ h_3^n(x) &= \int_0^x h_1^n(t) f_2(x-t)\,dt + \int_0^\infty h_2^n(t) f_2(x+t)\,dt + f_2(x). \end{aligned} \tag{3.1}$$

We now prove the following convergence theorem concerning the above approximation sequence.

THEOREM 3.1. *Let (h_1, h_2, h_3) be the unique solution of (2.5). Then $\sup_{x\in[0,\infty)} e^{-\sigma x} |h_i^n(x) - h_i(x)| \to 0$ as $n \to \infty$, for $i = 1, 2, 3$.*

PROOF. We first show that the sequence is monotonically increasing. For $n = 1$, we can see

$$\begin{aligned} h_1^1(x) &= \int_0^x \left(\int_0^\infty f_1(\xi) f_2(\xi + t)\,d\xi + f_2(t) \right) g_2(x-t)\,dt \ge h_1^0(x), \\ h_2^1(x) &= \int_0^x f_1(t) f_1(x-t)\,dt + \int_0^\infty h_1^1(t) f_1(x+t)\,dt + f_1(x) \ge h_2^0(x), \\ h_3^1(x) &= \int_0^x h_1^1(t) f_2(x-t)\,dt + \int_0^\infty h_2^1(t) f_2(x+t)\,dt + f_2(x) \ge h_3^0(x). \end{aligned}$$

Assuming that for some $n > 1$, $h_i^n \ge h_i^{n-1}$ $(i = 1, 2, 3)$, we have

$$\begin{aligned} h_1^{n+1}(x) - h_1^n(x) &= \int_0^x \left[h_3^n(t) - h_3^{n-1}(t)\right] g_2(x-t)\,dt \ge 0, \\ h_2^{n+1}(x) - h_2^n(x) &= \int_0^x \left[h_2^n(t) - h_2^{n-1}(t)\right] f_1(x-t)\,dt + \int_0^\infty \left[h_1^{n+1}(t) - h_1^n(t)\right] f_1(x+t)\,dt \ge 0, \\ h_3^{n+1}(x) - h_3^n(x) &= \int_0^x \left[h_1^{n+1}(t) - h_1^n(t)\right] f_2(x-t)\,dt + \int_0^\infty \left[h_2^{n+1}(t) - h_2^n(t)\right] f_2(x+t)\,dt \ge 0. \end{aligned}$$

We then show that $h_i^n(x)$ are bounded from above. To this end, we recall that existence and uniqueness of solutions to the following system of integral equations were established in [11] under the condition (2.10):

$$\begin{aligned} k_1(x) &= \int_0^x \eta k_3(t) e^{-\sigma(x-t)}\,dt, \\ k_2(x) &= \int_0^x \eta k_2(t) e^{-\sigma(x-t)}\,dt + \int_0^\infty \eta k_1(t) e^{-\sigma(x+t)}\,dt + \eta e^{-\sigma x}, \\ k_3(x) &= \int_0^x \eta k_1(t) e^{-\sigma(x-t)}\,dt + \int_0^\infty \eta k_2(t) e^{-\sigma(x+t)}\,dt + \eta e^{-\sigma x}. \end{aligned} \tag{3.2}$$

Furthermore, it was shown in [11] that for $i = 1, 2, 3$, $k_i(x)e^{-\sigma x}$ is integrable on $(0, \infty)$. Now we prove that $h_i^n(x) \le k_i(x)$ $(i = 1, 2, 3)$. By (2.1) and (2.10) $h_i^0(x) \le k_i(x)$ $(i = 1, 2, 3)$. Assuming that $h_i^n(x) \le k_i(x)$ $(i = 1, 2, 3)$, we find

$$h_1^{n+1}(x) - k_1(x) \le \int_0^x [h_3^n(t) - k_3(t)]\, g_2(x-t)\, dt \le 0,$$
$$h_2^{n+1}(x) - k_2(x) \le \int_0^x [h_2^n(t) - k_2(t)]\, f_1(x-t)\, dt + \int_0^\infty \left[h_1^{n+1}(t) - k_1(t)\right] f_1(x+t)\, dt \le 0,$$
$$h_3^{n+1}(x) - k_3(x) \le \int_0^x \left[h_1^{n+1}(t) - k_1(t)\right] f_2(x-t)\, dt + \int_0^\infty \left[h_2^{n+1}(t) - k_2(t)\right] f_2(x+t)\, dt \le 0.$$

Therefore, $(h_1^n,\, h_2^n,\, h_3^n)$ converges pointwise to (h_1, h_2, h_3) as $n \to \infty$. Furthermore, $|h_i^n(x) - h_i(x)|e^{-\sigma x}$ is integrable on $(0, \infty)$.

Letting $w_i^n(x) = |h_i^n(x) - h_i(x)|$ $(i = 1, 2, 3)$, then (w_1^n, w_2^n, w_3^n) satisfies the following:

$$\begin{aligned}
e^{-\sigma x} w_1^{n+1}(x) &\le \int_0^x \eta w_3^n(t) e^{-\sigma(2x-t)}\, dt \le \int_0^\infty \eta w_3^n(t) e^{-\sigma t}\, dt,\\
e^{-\sigma x} w_2^{n+1}(x) &\le \int_0^x \eta w_2^n(t) e^{-\sigma(2x-t)}\, dt + e^{-2\sigma x} \int_0^\infty \eta w_1^{n+1}(t) e^{-\sigma t}\, dt\\
&\le \int_0^\infty \eta w_2^n(t) e^{-\sigma t}\, dt + e^{-2\sigma x} \int_0^\infty \eta w_1^{n+1}(t) e^{-\sigma t}\, dt,\\
e^{-\sigma x} w_3^{n+1}(x) &\le \int_0^x \eta w_1^{n+1}(t) e^{-\sigma t}\, dt + e^{-2\sigma x} \int_0^\infty \eta w_2^{n+1}(t) e^{-\sigma t}\, dt\\
&\le \int_0^\infty \eta w_1^{n+1}(t) e^{-\sigma t}\, dt + e^{-2\sigma x} \int_0^\infty \eta w_2^{n+1}(t) e^{-\sigma t}\, dt.
\end{aligned}$$

Since $w_i^n(x)e^{-\sigma x}$ is integrable on $(0, \infty)$ and $w_i^n(x) \to 0$ pointwise as $n \to \infty$, by the dominated convergence theorem, we obtain that as $n \to \infty$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} e^{-\sigma x} w_1^{n+1}(x) &\le \int_0^\infty \eta w_3^n(t) e^{-\sigma t}\, dt \to 0,\\
\sup_{x\in[0,\infty)} e^{-\sigma x} w_2^{n+1}(x) &\le \int_0^\infty \eta w_2^n(t) e^{-\sigma t}\, dt + \int_0^\infty \eta w_1^{n+1}(t) e^{-\sigma t}\, dt \to 0,\\
\sup_{x\in[0,\infty)} e^{-\sigma x} w_3^{n+1}(x) &\le \int_0^\infty \eta w_1^{n+1}(t) e^{-\sigma t}\, dt + \int_0^\infty \eta w_2^{n+1}(t) e^{-\sigma t}\, dt \to 0.
\end{aligned}$$

This establishes the desired result. ∎

REMARK 3.2. The above result indicates that the sequence converges uniformly on any compact set $[0, b]$.

We now show additional properties of the solution to (2.5). The first one is concerned with the boundedness of the solution.

THEOREM 3.3. *The solution (h_1, h_2, h_3) of (2.5) is bounded on $[0, \infty)$. Furthermore, the sequence $\{h_1^n, h_2^n, h_3^n\}_{n=0}^\infty$ is bounded on $[0, \infty)$ uniformly in n.*

PROOF. Define

$$\begin{aligned}
\bar{H}_1(x) &= \int_0^x h_3(t) \bar{G}_1(x-t)\, dt,\\
\bar{H}_2(x) &= \int_0^x h_2(t) \bar{F}_1(x-t)\, dt + \int_0^\infty h_1(t) \bar{F}_1(x+t)\, dt + \bar{F}_1(x),\\
\bar{H}_3(x) &= \int_0^x h_1(t) \bar{F}_2(x-t)\, dt + \int_0^\infty h_2(t) \bar{F}_2(x+t)\, dt + \bar{F}_2(x).
\end{aligned} \tag{3.3}$$

By (2.1) and (2.5), we find

$$\frac{d}{dx}\bar{H}_2(x) = h_2(x) - \int_0^x h_2(t)f_1(x-t)\,dx - \int_0^\infty h_1(t)f_1(x+t)\,dt - f_1(x)$$
$$= h_2(x) - h_2(x) = 0.$$

Thus, $\bar{H}_2(x) = \bar{H}_2(0) = \int_0^\infty h_1(t)\bar{F}_1(t)\,dt + 1$. Using this fact and (2.5), we obtain the following inequality:

$$h_2(x) \le \eta\bar{H}_2(x) = \eta\bar{H}_2(0).$$

This establishes the boundedness of h_2 on $[0,\infty)$. To show that h_1 and h_3 are bounded, we add the first and third equations of (3.3) and differentiate to get

$$\frac{d}{dx}\left(\bar{H}_1(x) + \bar{H}_3(x)\right) = h_3(x) - \int_0^x h_3(t)g_2(x-t)\,dx$$
$$+ h_1(x) - \int_0^x h_1(t)f_2(x-t)\,dx - \int_0^\infty h_2(t)f_2(x+t)\,dt - f_2(x)$$
$$= h_3(x) - h_1(x) + h_1(x) - h_3(x) = 0.$$

Hence, $\bar{H}_1(x) + \bar{H}_2(x) = \bar{H}_3(0) = \int_0^\infty h_2(t)\bar{F}_2(t)\,dt + 1$. Proceeding analogously, we find that $h_1 + h_3$ is bounded on $[0,\infty)$. Since h_1 and h_3 both are nonnegative, each of them must be bounded.

To show that the sequence $\{h_1^n, h_2^n, h_3^n\}_{n=0}^\infty$ is uniformly bounded, we employ an argument as above. In particular, define a sequence $\{\bar{H}_1^n, \bar{H}_2^n, \bar{H}_3^n\}_{n=0}^\infty$ by

$$\bar{H}_1^n(x) = \int_0^x h_3^{n-1}(t)\bar{G}_1(x-t)\,dt,$$
$$\bar{H}_2^n(x) = \int_0^x h_2^{n-1}(t)\bar{F}_1(x-t)\,dt + \int_0^\infty h_1^n(t)\bar{F}_1(x+t)\,dt + \bar{F}_1(x),$$
$$\bar{H}_3^n(x) = \int_0^x h_1^n(t)\bar{F}_2(x-t)\,dt + \int_0^\infty h_2^n(t)\bar{F}_2(x+t)\,dt + \bar{F}_2(x).$$

It is not difficult to show that

$$\frac{d}{dx}\bar{H}_2^n(x) = h_2^{n-1}(x) - h_2^n(x) \le 0$$

by monotonicity of h_2^n. Hence, $\bar{H}_2^n(x) \le \bar{H}_2^n(0) \le \int_0^\infty h_1^n(x)e^{-\sigma x}\,dx + 1 \le C_1$. Since $h_2^n(x) \le \eta\bar{H}_2^n(x)$, we can see that $h_2^n(x)$ is uniformly bounded in n. Similarly, we have

$$\frac{d}{dx}\left(\bar{H}_1^n(x) + \bar{H}_3^n(x)\right) = h_3^{n-1}(x) - h_3^n(x) \le 0$$

by monotonicity of h_3^n. Hence, arguing as before we find that h_1^n and h_3^n are bounded on $[0,\infty)$, uniformly in n. ∎

We then establish the following asymptotic property for the solution.

THEOREM 3.4. *If* $\lim_{x\to\infty} h_3(x)$ *exists, then* $\lim_{x\to\infty} h_1(x)$ *exists and* $\lim_{x\to\infty} h_1(x) = \lim_{x\to\infty} h_3(x)$.

PROOF. Let $X_i(s)$ be the Laplace transform of $h_i(x)$. Taking the Laplace transform on both sides in the first equation of (2.5) and using the convolution property of the Laplace transform, we get

$$X_1(s) = D_2(s)X_3(s),$$

where $D_2(s)$ is the Laplace transform of $g_2(x)$. Applying the final-value theorem (e.g., [12, p. 83]) we obtain

$$\lim_{x\to\infty} h_1(x) = \lim_{s\to 0} sX_1(s) = \lim_{s\to 0} sD_2(s)X_3(s) = \lim_{s\to 0} D_2(s) \lim_{s\to 0} sX_3(s) = \lim_{x\to\infty} h_3(x).$$

This establishes the desired result. ∎

Finally, we prove that the solution is positive for all positive x. Our numerical results presented in the next section clearly demonstrate these three properties.

THEOREM 3.5. *Let (h_1, h_2, h_3) be the solution of (2.5). Then for any $x \in [0,\infty)$, $h_1(x) \geq \sigma^2 x e^{-\eta x}$, $h_2(x) \geq \sigma$, and $h_3(x) \geq \sigma e^{-\eta x}$. Moreover, $h_1(x) + h_3(x) \geq \sigma$ for all $x \in [0,\infty)$, and if $\lim_{x\to\infty} h_3(x)$ exists, then $\lim_{x\to\infty} h_3(x) = \lim_{x\to\infty} h_1(x) \geq \sigma/2$.*

PROOF. In view of (2.5) and (3.3), we have that $h_2(x) \geq \sigma \bar{H}_2(x) = \sigma \bar{H}_2(0) \geq \sigma$. Meanwhile, by the nonnegativity of all integrands in (2.5), we get from the third equation of (2.5)

$$h_3(x) = \int_0^x h_1(t) f_2(x-t)\, dt + \int_0^\infty h_2(t) f_2(x+t)\, dt + f_2(x) \geq f_2(x) \geq \sigma e^{-\eta x}.$$

Then the result for h_1 follows from the first equation of (2.5), since

$$h_1(x) \geq \int_0^x \sigma e^{-\eta t} g_2(x-t)\, dt \geq \int_0^x \sigma^2 e^{-\eta t} e^{-\eta(x-t)}\, dt = \sigma^2 x e^{-\eta x}.$$

Furthermore, we find that

$$h_1(x) + h_3(x) \geq \sigma \left(\bar{H}_1(x) + \bar{H}_3(x)\right) = \sigma \bar{H}_3(0) \geq \sigma.$$

If $\lim_{x\to\infty} h_3(x)$ exists, then from the previous theorem and the above inequality, we have $\lim_{x\to\infty} h_1(x) = \lim_{x\to\infty} h_3(x) \geq \sigma/2$. ∎

3.2. Discretization of the Monotone Sequence

In this section, we develop a discretization of the monotone sequence presented in the previous section. Note that all the integrals involved in the construction of the monotone sequence are convergent. Let $S^{\Delta x}(a, b, I)$ denote a standard numerical method for integrating a function $I(t)$ on the interval (a, b) with mesh size Δx. To integrate a function I on the infinite domain $(0,\infty)$, we use the following algorithm. Let B, $\Delta B > 0$ be constants and $\mathrm{TOL}_1 > 0$ be sufficiently small. Divide $[0, B]$ into m subdivisions with mesh size Δx (i.e., $m = B/\Delta x$). Compute $S^{\Delta x}(0, B, I)$ and $S^{\Delta x}(0, B + \Delta B, I)$. If the relative error satisfies

$$\left| \frac{S^{\Delta x}(0, B + \Delta B, I) - S^{\Delta x}(0, B, I)}{S^{\Delta x}(0, B, I)} \right| \leq \mathrm{TOL}_1,$$

then the integration over the domain $[0, B]$ is considered a good approximation to the integral of I over $(0,\infty)$. Otherwise, replace B by $B + \Delta B$ and repeat the procedure. Thus, during a simulation our domain of integration might change when computing each element of the monotone sequence h_i^n. Such an algorithm works well for the integrals involved in the system (2.5) due to the fact that all the integrands are positive functions for positive x (by Theorem 3.5). In what follows, we assume that at every iteration such a constant, for convenience we always denote it by B, has been found by the above algorithm.

Now we turn our attention to the numerical method for approximating the solution (h_1, h_2, h_3). Choose a sufficiently small positive number TOL_2, and for $j = 1, 2, \dots, m$, let $h_i^{n,j}$ be the numerical approximation of $h_i^n(x_j)$, $i = 1, 2, 3$. Let $f_i^j = f_i(x_j)$ and $g_2^j = g_2(x_j)$, $i = 1, 2$. Denote by $P(\mathbf{h}_i^n)$ the cubic interpolant of the vector $\mathbf{h}_i^n = [h_i^{n,1}, h_i^{n,2}, \dots, h_i^{n,m}]$ (e.g., [13, p. 143]), and for any function ϕ, denote by $\phi^{j+}(t) = \phi(x_j + t)$ and by $\phi^{j-}(t) = \phi(x_j - t)$. Compute

$$h_1^{0,j} = 0, \qquad h_2^{0,j} = f_1(x_j), \qquad h_3^{0,j} = S^{\Delta x}\left(0, B, f_1 f_2^{j+}\right) + f_2(x_j).$$

For $n = 1, 2, \dots$, compute

$$\begin{aligned}
h_1^{n,j} &= S^{\Delta x}\left(0, x_j, P\left(\mathbf{h}_3^{n-1}\right) g_2^{j-}\right),\\
h_2^{n,j} &= S^{\Delta x}\left(0, x_j, P\left(\mathbf{h}_2^{n-1}\right) f_1^{j-}\right) + S^{\Delta x}\left(0, B, P\left(\mathbf{h}_1^{n}\right) f_1^{j+}\right) + f_1^j,\\
h_3^{n,j} &= S^{\Delta x}\left(0, x_j, P\left(\mathbf{h}_1^{n}\right) f_2^{j-}\right) + S^{\Delta x}\left(0, B, P\left(\mathbf{h}_2^{n}\right) f_2^{j+}\right) + f_2^j.
\end{aligned}$$

If $\sup_{i,j} |h_i^{n,j} - h_i^{n-1,j}| \leq \mathrm{TOL}_2$, then $h_i^{n,j}$ is taken to be the approximation of the solution of (2.5). Note that in the computations presented here, Simpson's method is used for integration, with the starting values obtained from the trapezoidal rule.

4. NUMERICAL RESULTS

This section is devoted to using the algorithm described previously to study the behavior of the system (2.5). In our first example, we test the accuracy of the numerical method.

4.1. Testing the Method

Assume that all parameters $\lambda_i(x) = c$ and $\mu_i(x) = c$ (i.e., they are constants). Then differentiating (2.5) we obtain the following system of differential equations:

$$\begin{aligned} h_1' &= ch_3 - ch_1, & h_1(0) &= 0, \\ h_2' &= 0, & h_2(0) &= \int_0^\infty h_1(t) c e^{-ct}\, dt + c, \\ h_3' &= ch_1 - ch_3, & h_3(0) &= \int_0^\infty h_2(t) c e^{-ct} + c. \end{aligned}$$

A routine calculation shows that the solution of this system is given by

$$\begin{aligned} h_1 &= \frac{3}{2} c \left(1 - e^{-2cx}\right), \\ h_2 &= 2c, \\ h_3 &= \frac{3}{2} c \left(1 + e^{-2cx}\right). \end{aligned}$$

In Table 1, we present the difference between the computational and exact solutions, which is given by

$$e_i = \sup_{i,j} \left| h_i(x_j) - h_i^{n,j} \right|, \qquad i = 1, 2, 3.$$

In this simulation, $c = 4$ and $\mathrm{TOL}_1 = \mathrm{TOL}_2 = 10^{-5}$.

Table 1. Computational error for the constant parameter case.

Δx	1.00×10^{-1}
e_1	7.43×10^{-4}
e_2	7.34×10^{-4}
e_3	3.71×10^{-4}

4.2. An Example Satisfying Condition (2.10)

In this example, we choose the functions $\lambda_1 = 2 + 0.1 \sin(10x)$, $\lambda_2 = 3 + 0.2 \cos(15x)$, $\mu_1 = 3 + 0.2/(1 + x)$, and $\mu_2 = 2 + 0.4x/(1 + x^2)$. It is not too difficult to verify that these functions satisfy condition (2.10). The numerical solution of (2.5) for this choice of functions is presented in Figure 1.

4.3. Study of the Failure Frequency and Availability Relation

By straightforward calculations, it can be shown that for the constant parameter case discussed in Section 4.1, the stationary failure frequency of the system is related to the limiting availability of the system as follows:

$$W = A(a_1 + a_2),$$

where $a_1^{-1} = \int_0^\infty \bar{F}_1(x)\, dx$ and $a_2^{-1} = \int_0^\infty \bar{F}_2(x)\, dx$. The above relation is equivalent to the following equation:

$$\int_0^\infty \bar{H}_3(x) E_1(x) \bar{F}_1(x)\, dx = 0, \tag{4.1}$$

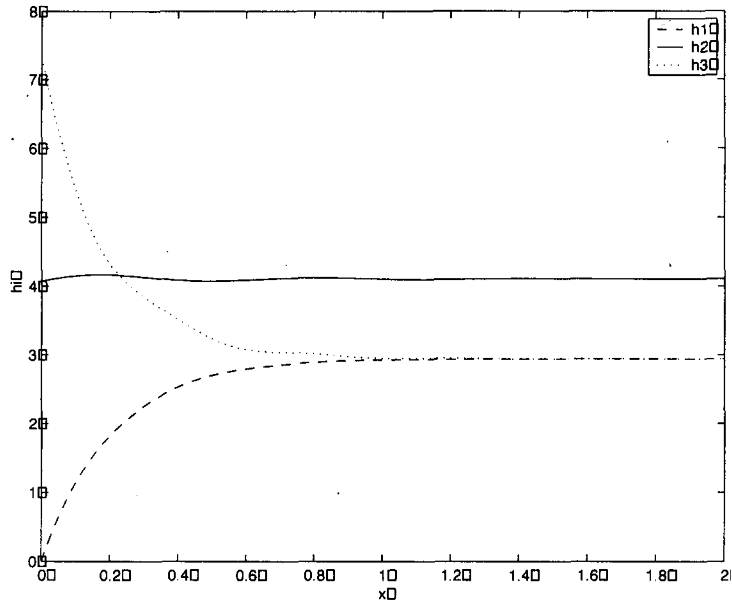

Figure 1. Numerical solution of system (2.5).

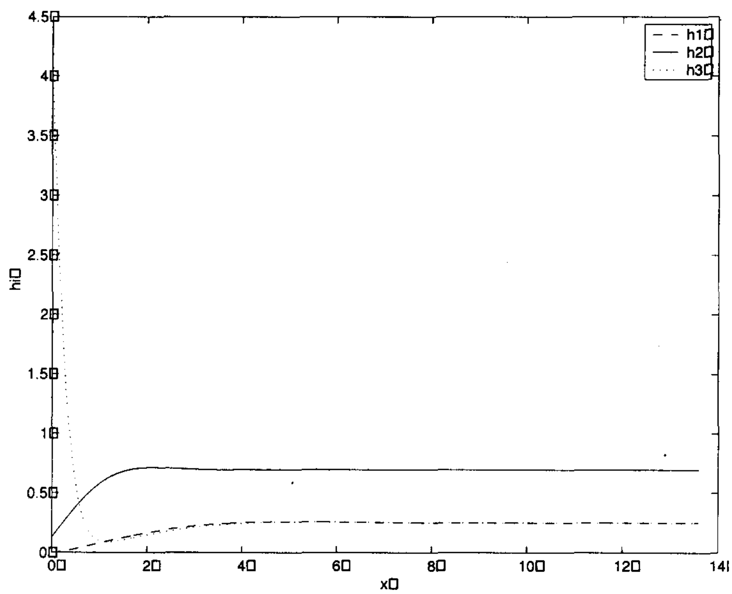

Figure 2. Numerical solution for system (2.5).

where

$$\bar{H}_3(x) = \int_0^x h_1(t)\bar{F}_2(x-t)\,dt + \int_0^\infty h_2(t)\bar{F}_2(x+t)\,dt + \bar{F}_2(x), \qquad E_1(x) = \frac{f_1(x)}{\bar{F}_1(x)} - a_1.$$

Our next result indicates that (4.1) may not always hold. Choosing the parameters $\lambda_1(x) = 0.6x$, $\lambda_2(x) = 4 + 2x$, $\mu_1 = 0.5x$, $\mu_2(x) = 0.1x$, we present the numerical solution in Figure 2. For this computation, $\int_0^\infty \bar{H}_3(x)E_1(x)\bar{F}_1(x)\,dx = -0.1048$.

4.4. Oscillatory Behavior of Solutions

Next we show numerically that the solution of (2.5) can exhibit oscillatory behavior even when the parameters λ_i and μ_i are nonoscillatory. In this example, we let $\lambda_1(x) = 0.6x^5$, $\lambda_2(x) = 2x^3$,

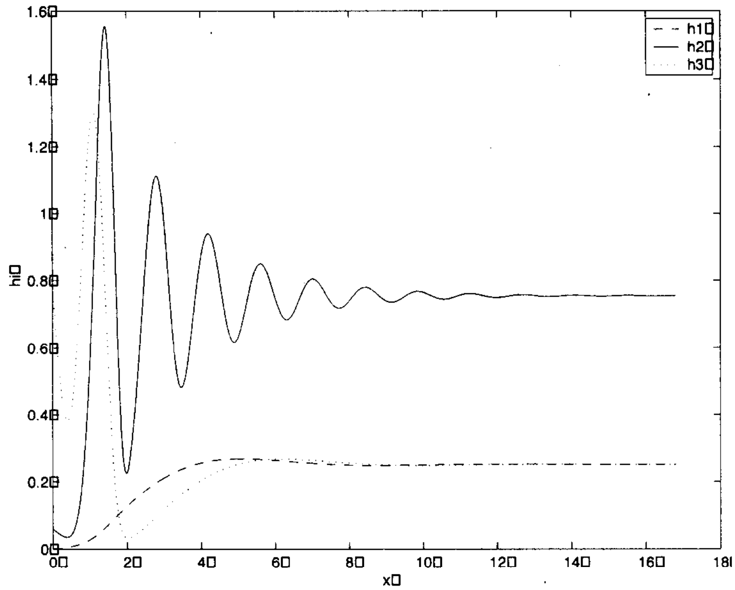

Figure 3. Oscillatory behavior of the numerical solution of (2.5).

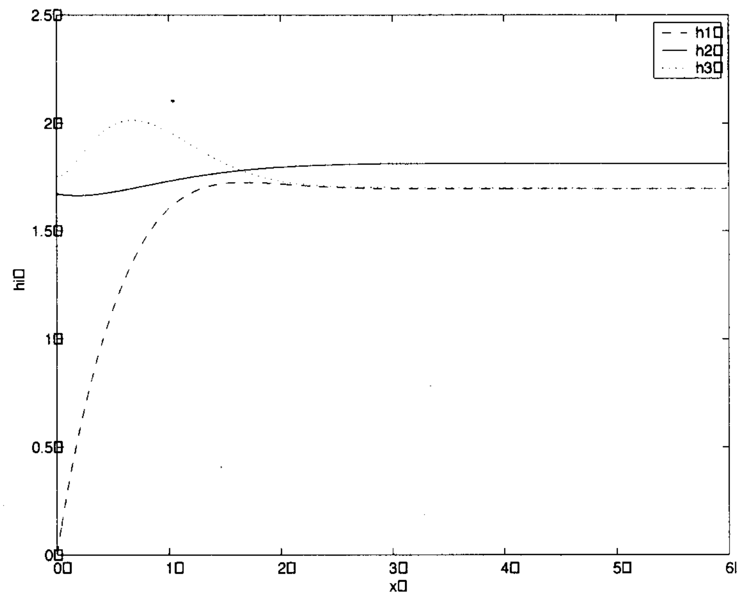

Figure 4. The numerical solution h_i, $i = 1, 2, 3$, for the Erlang and hyperexponential distributions.

$\mu_1(x) = 0.5x^4$, $\mu_2(x) = 0.1x$, and we present the numerical results in Figure 3. We remark that for this choice (4.1) is not valid as well, since $\int_0^\infty \bar{H}_3(x)E_1(x)\bar{F}_1(x)\,dx = -0.5451$.

4.5. Example Arising in Applications

Erlang and hyperexponential random variables are two important variables in the reliability problems. In fact, the models with Erlang uptime and hyperexponential repair time seem to be as general as those being realistically considered. The Erlang random uptime means that the machines will eventually fail after certain phases of the lifetime, and the hyperexponential random repair time means that the machines could be repaired by certain different methods with different time periods. These two variables are also considered in [14] for an unreliable machine schedule problem. Specifically, we assume here that the uptime of the machines is of Erlang distribution, i.e.,

$$F_i(t) = 1 - e^{-\gamma_i t} \sum_{k=0}^{n_i - 1} \frac{(\gamma_i t)^k}{k!},$$

where γ_i $(i = 1, 2)$ are constants. We assume that the repair time of the machines is of hyperexponential distribution, i.e.,

$$G_i(t) = \sum_{k=1}^{m_i} \alpha_{i,k} \left(1 - e^{-\beta_{i,k} t}\right),$$

where for $i = 1, 2$, $0 < \beta_{i,1} < \beta_{i,2} < \cdots < \beta_{i,m_i}$, $0 < \alpha_{i,k}$, and $\sum_{k=1}^{m_i} \alpha_{i,k} = 1$. From (2.1), one can see that $\lambda_i(t) = F_i'(t)/(1 - F_i(t))$ and $\mu_i(t) = G_i'(t)/(1 - G_i(t))$ for $i = 1, 2$. Standard

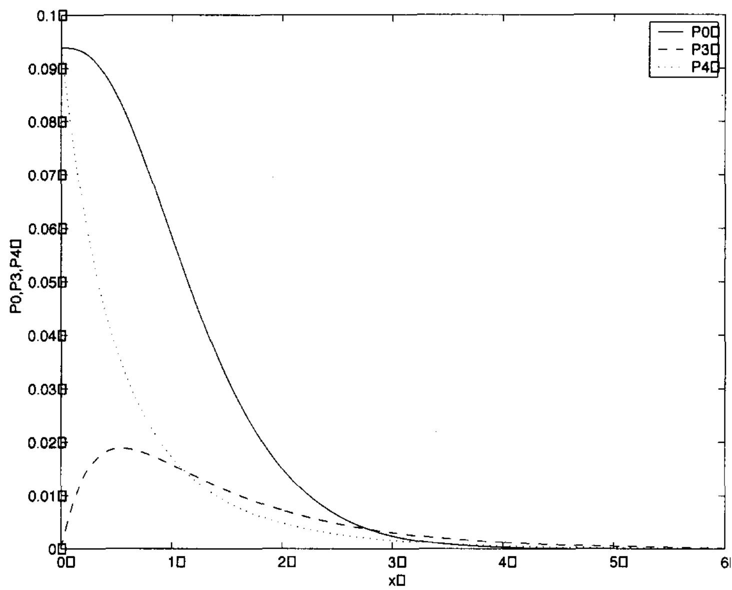

Figure 5. The numerical solutions for the functions P_0, P_3, and P_4.

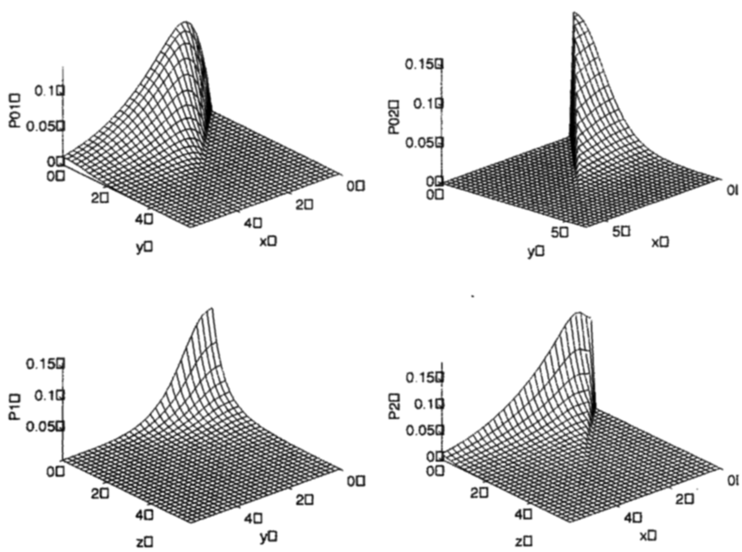

Figure 6. The numerical solutions for P_{01}, P_{02}, P_1, and P_2.

computations then give the following forms for λ_i and μ_i, $i = 1, 2$:

$$\lambda_i(t) = \frac{1}{(n_i - 1)!} \frac{\gamma_i^{n_i} t^{n_i - 1}}{\sum_{k=0}^{n_i - 1} (\gamma_i^k t^k / k!)}, \qquad \mu_i(t) = \frac{\sum_{k=1}^{m_i} \beta_{i,k} \alpha_{i,k} e^{-\beta_{i,k} t}}{\sum_{k=1}^{m_i} \alpha_{i,k} e^{-\beta_{i,k} t}}.$$

For our numerical results, we let $n_i = 3$ and $m_i = 3$, $i = 1, 2$. We choose $\gamma_1 = 1$, $\gamma_2 = 2$, $\beta_{i,1} = 1$, $\beta_{i,2} = 2$, $\beta_{i,3} = 3$, and $\alpha_{i,k} = 1/3$, $k = 1, 2, 3$. Hence, for this choice of parameters we have

$$\lambda_1(t) = \frac{t^2}{2 + 2t + t^2}, \qquad \lambda_2(t) = \frac{8t^2}{2 + 4t + 4t^2}, \qquad \mu_1(t) = \mu_2(t) = \frac{e^{2t} + 2e^t + 3}{e^{2t} + e^t + 1}.$$

The numerical solution of system (2.5) is presented in Figure 4. Furthermore, making use of (2.6), the numerical solution of system (2.2)–(2.4) is computed and presented in Figures 5 and 6.

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